Elementary Factors and Reduced Minors for Linear Systems over Commutative Rings

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Abstract. In 1994, Sule presented the necessary and sufficient conditions of the feedback stabilizability of systems over unique factorization domains in terms of elementary factors and in terms of reduced minors. Recently, Mori and Abe have generalized his theory over commutative rings. They have introduced the notion of the generalized elementary factor, which is a generalization of the elementary factor, and have given the necessary and sufficient condition of the feedback stabilizability. In this paper, we present two generalization of the reduced minors. Using each of them, we state the necessary and sufficient condition of the feedback stabilizability over commutative rings. Further we present the relationship between the generalizations and the generalized elementary factors.

Keywords. Linear systems, Feedback stabilization, Factorization approach, Systems over rings

1. Introduction. This paper is concerned with the coordinate-free approach to control systems. The coordinate-free approach is a factorization approach but does not require the coprime factorizations of plants.

The factorization approach was patterned after Desoer *et al.*[4] and Vidyasagar *et al.*[21], which has the advantage that it embraces, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, 1-D as well as n-D (multidimensional) systems, etc.[21]. In this approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. A transfer function of this approach is considered as the ratio of two stable causal transfer functions and the set of stable causal transfer functions forms a commutative ring.

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For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer matrices, which is satisfied in the case where the set of stable causal transfer functions is such a commutative ring as a Euclidean domain, a principal ideal, or a Bézout domain.

However, Anantharam in [1] showed that there exist models in which some stabilizable plants do not have right-/left-coprime factorizations. He considered the case where $\mathbb{Z}[\sqrt{5}\mathrm{i}]$ ($\simeq \mathbb{Z}[x]/(x^2+5)$) is the set of stable causal transfer functions, where $\mathbb{Z}$ is the ring of integers and i the imaginary unit. Using it, he showed that there exists a stabilizable plant which does not have right-/left-coprime factorizations. Further Mori in [14] has recently considered the case where $\mathbb{R}[z^2, z^3]$ is the set of stable causal transfer functions, where z denotes the unit delay operator and $\mathbb{R}$ the real field. This set is corresponding to the discrete finite-time delay system which does not have the unit delay. He has presented that in the model, some stabilizable plants do not have right-/left-coprime factorizations. Both $\mathbb{Z}[\sqrt{5}\mathrm{i}]$ and $\mathbb{R}[z^2, z^3]$ are not unique factorization domains.

Sule in [18, 19] has presented a theory of the feedback stabilization of multi-input multi-output strictly causal plants over commutative rings with some restrictions. This approach to the stabilization theory is called "coordinate-free approach" in the sense that the coprime factorizability of transfer matrices is not required.

In the case where the set of stable causal transfer functions is a unique factorization domain, Sule in [18] introduced two notions, that is, elementary factors and reduced minors. Using each of them he gave the necessary and sufficient condition of the feedback stabilizability of the causal plants over commutative rings (Theorem 4 and Corollary 2 of [18]). Especially, using elementary factors, Sule presented a construction method of a stabilizing controller of a stabilizable plant. Recently, Mori and Abe in [15, 16] have generalized his theory over commutative rings. They have introduced the notion of the generalized elementary factor, which is a generalization of the elementary factor, and have given the necessary and sufficient condition of the feedback stabilizability. Further Lin in [11] has presented the necessary and sufficient condition of the (structural) stabilizability of the multidimensional systems with the construction method of a stabilizing controller. In the case of the structural stability[5], it is known that the set of stable causal transfer functions is a unique factorization domain. Lin in [11] introduced a notion "generating polynomial" about the plants and presented the necessary and sufficient condition of the stabilizability of the multidimensional systems with the construction method of a stabilizing controller. It is known that the notion of the generating polynomial is equivalent to the notion of the reduced minors.

In this paper we have two main objectives. The first one is to generalize the notion of the reduced minors and, using the generalizations, to state the necessary and sufficient condition of the feedback stabilizability over commutative rings since the original definition has been given on unique factorization domains. We will present two generalizations. The other is to present the relationship between the generalizations and the generalized elementary factors.

Historically the minors concerning the plants are much investigated (e.g. [3, 8, 9, 10, 12, 22, 23, 24, 25]). We will present that in the coordinate-free approach, the minors can play a role to state the feedback stabilizability, that is, the *projectivity* of the ideal generated by minors concerning the plant is a criterion of the feedback stability.

This paper is organized as follows. After this introduction, we begin on the preliminary in Section 2, in which we give mathematical preliminaries, set up the feedback stabilization problem and present the previous results. In Section 3, we present the previous results of the feedback stabilizability expressed with the elementary factors, its derivation, and the reduced minors. We present a generalization of the reduced minor in Section 4 and using it present the necessary and sufficient condition of the feedback stabilizability over commutative rings in Section 5. Then in Section 6 we present another generalization of the reduced minors and its relation to the generalized elementary factors.

2. Preliminaries. In the following we begin by introducing the notations of commutative rings, matrices, and modules used in this paper. Then we give the formulation of the feedback stabilization problem.

2.1. Notations.

Commutative Rings. In this paper, we consider that any commutative ring has the identity 1 different from zero. Let $\mathcal{R}$ denote a (unspecified) commutative ring. The total ring of fractions of $\mathcal{R}$ is denoted by $\mathcal{F}(\mathcal{R})$.

We will consider that *the set of stable causal transfer functions* is a commutative ring, which is denoted by $\mathcal{A}$ throughout this paper. Further, we will use the following rings of fractions.

(i) The first one appears as the total ring of fractions of $\mathcal{A}$, which is denoted by $\mathcal{F}(\mathcal{A})$ or simply by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d$ is a nonzerodivisor$\}$. This will be considered as *the set of all possible transfer functions*.

(ii) Let f denote a nonzero (but possibly nonzerodivisor) element of $\mathcal{A}$. Given a set $S_f = \{1, f, f^2, \ldots\}$, which is a multiplicative subset of $\mathcal{A}$, we denote by $\mathcal{A}_f$

the ring of fractions of $\mathcal{A}$ with respect to the multiplicative subset S_f; that is, $\mathcal{A}_f = \{n/d \,|\, n \in \mathcal{A},\ d \in S_f\}$.

(iii) Let p denote a prime ideal of $\mathcal{A}$ and S the complement of the prime ideal p, that is, $S = \mathcal{A}\backslash\mathrm{p}$. Then S is a multiplicative subset of $\mathcal{A}$. We denote by $\mathcal{A}_{\mathrm{p}}$ the ring of fractions of $\mathcal{A}$ with respect to the multiplicative subset S; that is, $\mathcal{A}_{\mathrm{p}} = \{n/d \,|\, n \in \mathcal{A},\ d \in S\}$.

(iv) The last one is the total ring of fractions of $\mathcal{A}_f$ or $\mathcal{A}_{\mathrm{p}}$, which is denoted by $\mathcal{F}(\mathcal{A}_f)$ and $\mathcal{F}(\mathcal{A}_{\mathrm{p}})$; that is, $\mathcal{F}(\mathcal{A}_f) = \{n/d \,|\, n, d \in \mathcal{A}_f,\ d$ is a nonzerodivisor of $\mathcal{A}_f\}$ and $\mathcal{F}(\mathcal{A}_{\mathrm{p}}) = \{n/d \,|\, n, d \in \mathcal{A}_{\mathrm{p}},\ d$ is a nonzerodivisor of $\mathcal{A}_{\mathrm{p}}\}$. If f is a nonzero-divisor of $\mathcal{A}$, $\mathcal{F}(\mathcal{A}_f)$ coincides with the total ring of fractions of $\mathcal{A}$. Otherwise, they do not coincide.

In the case where $\mathcal{A}$ is a unique factorization domain, we call a in $\mathcal{A}$ *the radical of* b in $\mathcal{A}$ if a has all nonunit factors of b and is squarefree, that is, a does not have duplicated nonunit factors. Note here that the radical defined here is unique up to any unit multiple.

For convenience, throughout the paper, if $a \in \mathcal{A}$ ($a \in \mathcal{R}$), then a itself denotes $a/1$ in $\mathcal{A}_f$ and $\mathcal{A}_{\mathrm{p}}$ ($a/1$ in $\mathcal{F}(\mathcal{R})$). Moreover if $a \in \mathcal{A}_f$ or $\mathcal{A}_{\mathrm{p}}$ ($a \in \mathcal{R}$) and if there exists $b \in \mathcal{A}$ such that $a = b/1$ over $\mathcal{A}_f$ or $\mathcal{A}_{\mathrm{p}}$ (over $\mathcal{F}(\mathcal{R})$), then we regard a as an element of $\mathcal{A}$ ($\mathcal{R}$).

In the rest of the paper, we will use $\mathcal{R}$ as an unspecified commutative ring and mainly suppose that $\mathcal{R}$ denotes one of $\mathcal{A}$, $\mathcal{A}_f$, and $\mathcal{A}_{\mathrm{p}}$.

We will denote by $\mathrm{Spec}(\mathcal{R})$ the set of all prime ideals of $\mathcal{R}$ and by $\mathrm{Max}(\mathcal{R})$ the set of all maximal ideals of $\mathcal{R}$. Suppose that $\mathfrak{a}$ is an ideal of $\mathcal{R}$. Then we denote by $\mathfrak{a}_f$ the ideal of fractions of $\mathfrak{a}$ with respect to $\{1, f, f^2, \ldots\}$ with $f \in \mathcal{R}$ (that is, $\mathfrak{a}_f = \{n/d \,|\, n \in \mathfrak{a},\ d \in \{1, f, f^2, \ldots\}\}$) and by $\mathfrak{a}_{\mathrm{p}}$ the ideal of fractions of $\mathfrak{a}$ with respect to $\mathcal{R}\backslash\mathrm{p}$ with $\mathrm{p} \in \mathrm{Spec}(\mathcal{R})$ (that is, $\mathfrak{a}_{\mathrm{p}} = \{n/d \,|\, n \in \mathfrak{a},\ d \in \mathcal{R}\backslash\mathrm{p}\}$). If $\mathfrak{a}$ is an ideal of $\mathcal{R}$ and if S is a subset of $\mathcal{R}$, then we denote by $(\mathfrak{a} : S)$ the *quotient ideal* which is the set $\{f \in \mathcal{R} \,|\, fS \subset \mathfrak{a}\}$.

The reader is referred to Chapter 3 of [2] for the ring of fractions.

Matrices. The set of matrices over $\mathcal{R}$ of size $x \times y$ is denoted by $\mathcal{R}^{x \times y}$. Further, the set of square matrices over $\mathcal{R}$ of size x is denoted by $(\mathcal{R})_x$. The identity and the zero matrices are denoted by E_x and $O_{x \times y}$, respectively, if the sizes are required, otherwise they are denoted by E and O.

Matrix A over $\mathcal{R}$ is said to be *nonsingular* (*singular*) *over* $\mathcal{R}$ if the determinant of the matrix A is a nonzerodivisor (a zerodivisor) of $\mathcal{R}$. Matrices A and B over $\mathcal{R}$ are *right-* (*left-*)*coprime over* $\mathcal{R}$ if there exist matrices X and Y over $\mathcal{R}$

such that $XA + YB = E$ $(AX + BY = E)$ holds. Note that, in the sense of the above definition, two matrices which have no common right-(left-)factors except invertible matrices may not be right-(left-)coprime over $\mathcal{R}$. Further, an ordered pair (N, D) of matrices N and D is said to be a ***right-coprime factorization over*** $\mathcal{R}$ of P if (i) D is nonsingular over $\mathcal{R}$, (ii) $P = ND^{-1}$ over $\mathcal{F}(\mathcal{R})$, and (iii) N and D are right-coprime over $\mathcal{R}$. As the parallel notion, the *left-coprime factorization over* $\mathcal{R}$ of P is defined analogously. That is, an ordered pair $(\widetilde{D}, \widetilde{N})$ of matrices $\widetilde{N}$ and $\widetilde{D}$ is said to be a *left-coprime factorization over* $\mathcal{R}$ of P if (i) $\widetilde{D}$ is nonsingular over $\mathcal{R}$, (ii) $P = \widetilde{D}^{-1}\widetilde{N}$ over $\mathcal{F}(\mathcal{R})$, and (iii) $\widetilde{N}$ and $\widetilde{D}$ are left-coprime over $\mathcal{R}$. Note that the order of the "denominator" and "numerator" matrices is interchanged in the latter case. This is to reinforce the point that if (N, D) is a right-coprime factorization over $\mathcal{R}$ of P, then $P = ND^{-1}$, whereas if $(\widetilde{D}, \widetilde{N})$ is a left-coprime factorization over $\mathcal{R}$ of P, then $P = \widetilde{D}^{-1}\widetilde{N}$ according to [20]. For short, we may omit "over $\mathcal{R}$" when $\mathcal{R} = \mathcal{A}$, and "right" and "left" when the size of matrix is 1×1. In the case where matrices are potentially used to express *left* fractional form and/or *left* coprimeness, we usually attach a tilde '$\widetilde{\ }$' to symbols; for example $\widetilde{N}$, $\widetilde{D}$ for $P = \widetilde{D}^{-1}\widetilde{N}$ and $\widetilde{Y}$, $\widetilde{X}$ for $\widetilde{Y}N + \widetilde{X}D = E$.

Modules. Let $M_r(X)$ $(M_c(X))$ denote the $\mathcal{R}$-module generated by rows (columns) of a matrix X over $\mathcal{R}$. Let $X = AB^{-1} = \widetilde{B}^{-1}\widetilde{A}$ be a matrix over $\mathcal{F}(\mathcal{R})$, where A, B, $\widetilde{A}$, $\widetilde{B}$ are matrices over $\mathcal{R}$. It is known that $M_r([\,A^t \quad B^t\,]^t)$ $(M_c([\,\widetilde{A} \quad \widetilde{B}\,]))$ is unique up to an isomorphism with respect to any choice of fractions AB^{-1} of X $(\widetilde{B}^{-1}\widetilde{A}$ of $X)$ (Lemma 2.1 of [15]). Therefore, for a matrix X over $\mathcal{R}$, we denote by $\mathcal{T}_{X,\mathcal{R}}$ and $\mathcal{W}_{X,\mathcal{R}}$ the modules $M_r([\,A^t \quad B^t\,]^t)$ and $M_c([\,\widetilde{A} \quad \widetilde{B}\,])$, respectively.

An $\mathcal{R}$-module M is called *free* if it has a basis, that is, a linearly independent system of generators. The *rank* of a free $\mathcal{R}$-module M is equal to the cardinality of a basis of M, which is independent of the basis chosen. An $\mathcal{R}$-module M is called *projective* if it is a direct summand of a free $\mathcal{R}$-module, that is, there is a module N such that $M \oplus N$ is free. The reader is referred to Chapter 2 of [2] for the module theory.

We will consider occasionally ideals as modules in this paper. So, we will apply the words "projective," "free," and "isomorphic" to ideals. It is easy to check that an ideal which is free as a module is equivalent to a principal ideal whose generator is a nonzerodivisor.

2.2. Feedback Stabilization Problem. The stabilization problem considered in this paper follows that of Sule in [18], and Mori and Abe in [15], who consider the feedback system Σ [20, Ch.5, Figure 5.1] as in Figure 2.1. For further details the reader is referred to [20]. Throughout the paper, the plant we consider has m

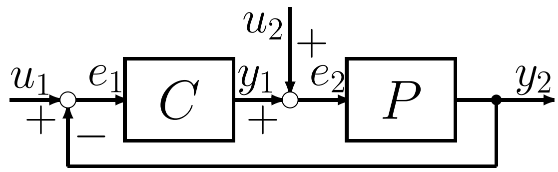

FIG. 2.1. *Feedback system* Σ.

inputs and n outputs, and its transfer matrix, which is also called a *plant* itself simply, is denoted by P and belongs to $\mathcal{F}^{n\times m}$. We can always represent P in the form of a fraction $P = ND^{-1}$ ($P = \widetilde{D}^{-1}\widetilde{N}$), where $N \in \mathcal{A}^{n\times m}$ ($\widetilde{N} \in \mathcal{A}^{n\times m}$) and $D \in (\mathcal{A})_m$ ($\widetilde{D} \in (\mathcal{A})_n$) with nonsingular D ($\widetilde{D}$).

DEFINITION 2.1. *For $P \in \mathcal{F}^{n\times m}$ and $C \in \mathcal{F}^{m\times n}$, a matrix $H(P,C) \in (\mathcal{F})_{m+n}$ is defined as*

$$H(P,C) = \begin{bmatrix} (E_n + PC)^{-1} & -P(E_m + CP)^{-1} \\ C(E_n + PC)^{-1} & (E_m + CP)^{-1} \end{bmatrix} \tag{2.1}$$

provided that $\det(E_n + PC)$ *is a nonzerodivisor of* $\mathcal{A}$*. This* $H(P,C)$ *is the transfer matrix from* $[\, u_1^t \quad u_2^t \,]^t$ *to* $[\, e_1^t \quad e_2^t \,]^t$ *of the feedback system* Σ*. If (i)* $\det(E_n + PC)$ *is a nonzerodivisor of* $\mathcal{A}$ *and (ii)* $H(P,C) \in (\mathcal{A})_{m+n}$*, then we say that the plant* P *is* stabilizable, P *is* stabilized *by* C*, and* C *is a* stabilizing controller *of* P.

Since the transfer matrix $H(P,C)$ of the stable causal feedback system has all entries in $\mathcal{A}$, we call the above notion $\mathcal{A}$-*stabilizability*. One can further introduce the notion of $\mathcal{R}$-*stabilizability* with either $\mathcal{R} = \mathcal{A}_f$ or $= \mathcal{A}_{\mathrm{p}}$ as follows.

DEFINITION 2.2. *Suppose that* $\mathcal{R}$ *is either* $\mathcal{A}_f$ *with* $f \in \mathcal{A}\backslash\{0\}$ *or* $\mathcal{A}_{\mathrm{p}}$ *with* $\mathrm{p} \in \mathrm{Spec}(\mathcal{A})$*. If (i)* $\det(E_n + PC)$ *is a nonzerodivisor of* $\mathcal{R}$ *and (ii)* $H(P,C) \in (\mathcal{R})_{m+n}$*, then we say that the plant* P *is* $\mathcal{R}$-stabilizable*,* P *is* $\mathcal{R}$-stabilized *by* C*, and* C *is an* $\mathcal{R}$-stabilizing controller *of* P.

The causality of transfer functions is an important physical constraint. We employ, in this paper, the definition of the causality from Vidyasagar *et al.*[21, Definition 3.1].

DEFINITION 2.3. *Let* $\mathcal{Z}$ *be a prime ideal of* $\mathcal{A}$*, with* $\mathcal{Z} \neq \mathcal{A}$*, including all zerodivisors. Define the subsets* $\mathcal{P}$ *and* $\mathcal{P}_S$ *of* $\mathcal{F}$ *as follows:*

$$\mathcal{P} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{A},\ b \in \mathcal{A}\backslash\mathcal{Z}\},$$
$$\mathcal{P}_S = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{Z},\ b \in \mathcal{A}\backslash\mathcal{Z}\}.$$

Then every transfer function in $\mathcal{P}$ $(\mathcal{P}_S)$ *is called* causal (strictly causal). *Analogously, if every entry of a transfer matrix* F *is in* $\mathcal{P}$ $(\mathcal{P}_S)$*, the transfer matrix* F *is called* causal (strictly causal). *A matrix over* $\mathcal{A}$ *is said to be* $\mathcal{Z}$-nonsingular *if the determinant is in* $\mathcal{A}\backslash\mathcal{Z}$*, and* $\mathcal{Z}$-singular *otherwise.*

Before proceeding the next section, we here introduce several symbols used throughout this paper. The symbol $\mathcal{I}$ denotes the family of all sets of m distinct integers between 1 and $m+n$, and $\mathcal{J}$ the family of all sets of n distinct integers between 1 and $m+n$ (recall that m and n are the numbers of the inputs and the outputs, respectively). Normally, elements of $\mathcal{I}$ $(\mathcal{J})$ will be denoted by I (J) possibly with suffices. They will be used as suffices as well as sets. If I is an element of $\mathcal{I}$ and if $i_1, \ldots, i_m$ are elements of I with ascending order, that is, $i_a < i_b$ if $a < b$, then the symbol Δ_I denotes the $m \times (m+n)$ matrix whose (k, i_k)-entry is 1 for $i_k \in I$ and zero otherwise. Analogously if J is an element of $\mathcal{J}$ and if $j_1, \ldots, j_n$ are elements of J with ascending order, then the symbol Δ_J denotes the $n \times (m+n)$ matrix whose (k, j_k)-entry is 1 for $j_k \in J$ and zero otherwise.

3. Previous Results. In this section, we recall the previous results about the necessary and sufficient condition of the feedback stabilizability. First one is stated in terms of the elementary factors and the other in terms of the reduced minors.

3.1. Feedback Stabilizability in terms of Elementary Factors. To state the result, we first recall the notion of the elementary factors, which was defined under the assumption that $\mathcal{A}$ is a unique factorization domain.

DEFINITION 3.1. (Elementary Factors, [18, p.1689]) *Suppose that* $\mathcal{A}$ *is a unique factorization domain. Denote by* T *and* W *the matrices* $[\, N^t \quad dE_m \,]^t$ *and* $[\, N \quad dE_n \,]^t$ *over* $\mathcal{A}$ *with* $P = Nd^{-1}$*. Further denote by* $\mathcal{I}^*$ $(\mathcal{J}^*)$ *the set of* I*'s in* $\mathcal{I}$ $(J$*'s in* $\mathcal{J})$ *such that* $\Delta_I T$ $(\Delta_J W^t)$ *is nonsingular. Then for each* $I \in \mathcal{I}^*$*, let* f_I *be the radical of the least common multiple of all the denominators of the matrix* $T(\Delta_I T)^{-1}$ *and for each* $J \in \mathcal{J}^*$*,* g_J *be the radical of the least common multiple of all the denominators of the matrix* $W^t(\Delta_J W^t)^{-1}$*. Then* f_I (g_J) *is called* the elementary factor of the matrix T (W) with respect to $I \in \mathcal{I}$ $(J \in \mathcal{J})$, $F = \{f_I \,|\, I \in \mathcal{I}^*\}$ the family of elementary factors of the matrix T, $G = \{g_J \,|\, J \in \mathcal{J}^*\}$ the family of elementary factors of the matrix W*, and* $H = \{h_{IJ} := f_I g_J \,|\, I \in \mathcal{I}^*, J \in \mathcal{J}^*\}$ the family of elementary factors of P.

Then the necessary and sufficient condition of the feedback stabilizability is given as follows.

THEOREM 3.2. (Theorem 4 of [18]) *Suppose that* $\mathcal{A}$ *is a unique factorization*

domain. Then the plant P is stabilizable if and only if the elementary factors of P are coprime, that is, $\sum_{I\in\mathcal{I}^*,J\in\mathcal{J}^*}(h_{IJ}) = \mathcal{A}$.

In the proof of this theorem, Sule gave a method to construct a stabilizing controller of the plant.

The result above has been extended to include systems over commutative rings by Mori and Abe in [16] as follows. They introduced the notion of the generalized elementary factors, which is a generalization of the elementary factors, and using it, stated the necessary and sufficient conditions of the feedback stabilizability over commutative rings.

DEFINITION 3.3. (Generalized Elementary Factors, Definition 3.1 of [16]) *Denote by T the matrix $[\, N^t \quad D^t \,]^t$ over $\mathcal{A}$ with $P = ND^{-1}$. For each $I \in \mathcal{I}$, an ideal Λ_{PI} over $\mathcal{A}$ is defined as*

$$\Lambda_{PI} = \{\lambda \in \mathcal{A} \,|\, \exists K \in \mathcal{A}^{(m+n)\times m}\ \lambda T = K\Delta_I T\}.$$

We call the ideal Λ_{PI} the generalized elementary factor *of the plant P with respect to I. Further, the set of all Λ_{PI}'s is denoted by $\mathcal{L}_P$, that is, $\mathcal{L}_P = \{\Lambda_{PI} \,|\, I \in \mathcal{I}\}$.*

In the case where $\mathcal{A}$ is a unique factorization domain, a generalized elementary factor with respect to $I \in \mathcal{I}$ is a principal ideal and the radical of its generator is an elementary factor of T with respect to I up to a unit multiple.

It is known that the generalized elementary factor of a plant P is independent of the choice of fractions $ND^{-1} = P$ (Lemma 3.3 of [16]).

The following is the necessary and sufficient conditions of the feedback stabilizability.

THEOREM 3.4. (Theorem 3.2 of [16]) *Consider a causal plant P. Then the following statements are equivalent:*

(i) *The plant P is stabilizable.*

(ii) *$\mathcal{A}$-modules $\mathcal{T}_{P,\mathcal{A}}$ and $\mathcal{W}_{P,\mathcal{A}}$ are projective.*

(iii) *The set of all generalized elementary factors of P generates $\mathcal{A}$; that is, $\mathcal{L}_P$ satisfies:*

(3.1)
$$\sum_{\Lambda_{PI}\in\mathcal{L}_P} \Lambda_{PI} = \mathcal{A}.$$

Provided that we can check (3.1) and that we can construct the right-coprime factorizations over $\mathcal{A}_{\lambda_I}$ of the given causal plant, where λ_I is a nonzero element of $\mathcal{A}$, Mori and Abe[16] have given a method to construct a causal stabilizing controller of a causal stabilizable plant, which has been given in the proof of "(iii)$\rightarrow$(i)" of Theorem 3.2 of [16].

3.2. Feedback Stabilizability in terms of Reduced Minors. We first recall the definition of the reduced minors and then state the necessary and sufficient conditions of the feedback stabilizability in terms of the reduced minors. We suppose in this subsection that $\mathcal{A}$ is a unique factorization domain.

DEFINITION 3.5. (Reduced Minors, [18, p.1690]) *Let P be a plant of $\mathcal{F}^{n\times m}$, N a matrix of $\mathcal{A}^{n\times m}$, and d an element of $\mathcal{A}$ such that $P = Nd^{-1}$. Denote by T and W the matrices $[\, N^t \quad dE_m \,]^t$ and $[\, N \quad dE_n \,]$. Let $t_I = \det(\Delta_I T)$ ($w_J = \det(\Delta_J W^t)$), which is a full-size minor of the matrix T (W), for $I \in \mathcal{I}$ ($J \in \mathcal{J}$). Let d_t (d_w) be the greatest common factor of t_I's (w_J's) and $a_I = t_I/d_t$ for $I \in \mathcal{I}$ ($b_J = w_J/d_w$ for $J \in \mathcal{J}$). Then a_I (b_J) is called the* reduced minor of the matrix T (W) with respect to $I \in \mathcal{I}$ ($J \in \mathcal{J}$), *the set* $\{a_I \mid I \in \mathcal{I}\}$ ($\{b_J \mid J \in \mathcal{J}\}$) the family of reduced minors of T (W).

It is known that the families of reduced minors of T and of W are identical modulo units (Lemma 5 of [18]).

Now, Corollary 2 of [18] including its comments can be stated as follows:

THEOREM 3.6. (cf. Corollary 2 of [18]) *Suppose that $\mathcal{A}$ is a unique factorization domain. A plant $P \in \mathcal{F}^{m\times n}$ is stabilizable if and only if the family of the reduced minors of T (and also of W) generates $\mathcal{A}$.*

The theorem above can be rewritten directly as follows.

COROLLARY 3.7. *Let t_I and w_J be as in Definition3.5. Then the following are equivalent:*

(i) *A plant $P \in \mathcal{F}^{m\times n}$ is stabilizable.*

(ii) *The ideal $\sum_{I\in\mathcal{I}}(t_I)$ is principal, or equivalently free as an $\mathcal{A}$-module.*

(iii) *The ideal $\sum_{J\in\mathcal{J}}(w_J)$ is principal, or equivalently free as an $\mathcal{A}$-module.*

4. Full-Size Minor Ideal. On the statements concerning the elementary factors and the reduced minors in Subsections 3.1 and 3.2, we have considered that the denominator matrices of the plant is expressed as dE_m or dE_n rather than general nonsingular matrices. This may be considered as a restriction on the expression of the plant. Thus we rather consider that P is expressed as either $P = ND^{-1}$ with $N \in \mathcal{A}^{n\times m}$ and $D \in (\mathcal{A})_m$ or $P = \widetilde{D}^{-1}\widetilde{N}$ with $\widetilde{N} \in \mathcal{A}^{n\times m}$ and $\widetilde{D} \in (\mathcal{A})_n$. Now we redefine the matrices T, W as $T = [\, N^t \quad D^t \,]^t$ and $W = [\, \widetilde{N} \quad \widetilde{D} \,]$. Further we consider that t_I's and w_J's are defined with the matrices T and W here. In the rest of this paper, we will use these notations unless otherwise stated.

We now introduce a notion to state the feedback stabilizability over commutative rings.

DEFINITION 4.1. (Full-Size Minor Ideals) *The ideal generated by* t_I*'s for* $I \in \mathcal{I}$ *is called the* full-size minor ideal *of the plant* P*. We denote it by* $\sum_{I\in\mathcal{I}}(t_I)$ *or simply* $\mathfrak{t}$*.*

We can also consider the ideal generated by w_J's for $J \in \mathcal{J}$, denoted by $\sum_{J\in\mathcal{J}}(w_J)$ or simply $\mathfrak{w}$. The ideals $\mathfrak{t}$ and $\mathfrak{w}$ depend on the fractional representation of the plant $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$. However, this is *not* a problem from the following reason. To state the feedback stabilizability in terms of the full-size minor ideals, we will regard them as modules. Further, when these ideals are considered as modules, both the ideals $\mathfrak{t}$ and $\mathfrak{w}$ are uniquely determined as modules up to isomorphism with respect to any choice of fractions ND^{-1} and $\widetilde{N}^{-1}\widetilde{D}$ of P as shown below.

LEMMA 4.2. *Let* P *be in* $\mathcal{F}(\mathcal{R})^{n\times m}$*, where* $\mathcal{R}$ *is one of* $\mathcal{A}$*,* $\mathcal{A}_f$ *with a nonzero* $f \in \mathcal{A}$*, and* $\mathcal{A}_{\mathrm{p}}$ *with a prime ideal* p *in* $\mathrm{Spec}(\mathcal{A})$*. For* $x = 1, 2$ *let* N_x*,* D_x*,* $\widetilde{N}_x$*,* $\widetilde{D}_x$ *be matrices over* $\mathcal{R}$ *with* $P = N_x D_x^{-1} = \widetilde{D}_x^{-1}\widetilde{N}_x$ *over* $\mathcal{F}(\mathcal{R})$*,* $T_x = [\, N_x^t \quad D_x^t \,]^t$ *and* $W_x = [\, \widetilde{N}_x \quad \widetilde{D}_x \,]$*. Further for* $x = 1, 2$ *and for* $I \in \mathcal{I}$*,* $J \in \mathcal{J}$*, let* $t_{xI} = \det(\Delta_I T_x)$*, and* $w_{xJ} = \det(\Delta_J W_x^t)$*. Then the ideals* $\sum_{I\in\mathcal{I}}(t_{1I})$*,* $\sum_{I\in\mathcal{I}}(t_{2I})$*,* $\sum_{J\in\mathcal{J}}(w_{1J})$*, and* $\sum_{J\in\mathcal{J}}(w_{2J})$ *are isomorphic to one another as* $\mathcal{R}$*-modules.*

Proof. We show first (i) $\sum_{I\in\mathcal{I}}(t_{1I}) \simeq \sum_{I\in\mathcal{I}}(t_{2I})$ and then (ii) $\sum_{I\in\mathcal{I}}(t_{1I}) \simeq \sum_{J\in\mathcal{J}}(w_{1J})$. The isomorphism $\sum_{J\in\mathcal{J}}(w_{1J}) \simeq \sum_{J\in\mathcal{J}}(w_{2J})$ can be proved analogously to (i) and so is omitted.

(i). Observe that in the case where $[\, N_2^t \quad D_2^t \,]^t = [\, N_1^t \quad D_1^t \,]^t X$ holds with some nonsingular matrix X over $\mathcal{R}$, the statement of the lemma obviously holds. Hence by considering $[\, N_2^t \quad D_2^t \,]^t \operatorname{adj}(D_2)$ as $[\, N_2^t \quad D_2^t \,]^t$, we can assume without loss of generality that D_2 is expressed as $d_2 E_m$ with nonzero d_2. Observe now that $[\, N_1^t \quad D_1^t \,]^t d_2 = [\, N_2^t \quad d_2 E_m \,] D_1$ holds. From this relation and the first observation, we now have (i).

(ii). It is sufficient to consider the case $P = Nd^{-1}$ with $N \in \mathcal{R}^{n\times m}$ and $d \in \mathcal{R}$ as in (i). In the case $P = ND^{-1}$, one can consider $P = (N \operatorname{adj}(D)) \det(D)^{-1}$.

First we define a bijective mapping τ from $\mathcal{I}$ to $\mathcal{J}$. For convenience we decompose I into I_N and I_d as follows

$$I_N = \{i \,|\, i \le n, i \in I\}, \;\; I_d = \{i \,|\, i > n, i \in I\}.$$

Corresponding to I_N and I_d, we define J_N and J_d as

$$J_N = [1, m] \backslash \{i - n \,|\, i \in I_d\}, \;\; J_d = \{i + m \,|\, i \in [1, n] \backslash I_N\}.$$

We now define the mapping $\tau : \mathcal{I} \to \mathcal{J}$ as

$$\tau : I_N \cup I_d \mapsto J_N \cup J_d.$$

Since I_N and I_d can be expressed by J_N and J_d as $I_N = [1, n] \backslash \{j - m \,|\, j \in J_d\}$, $I_d = \{j + n \,|\, j \in [1, m] \backslash J_N\}$, the inverse mapping $\tau^{-1} : \mathcal{J} \to \mathcal{I}$ can be defined naturally. Hence, the map τ is bijective.

Now let $T = [\, N^t \quad dE_m \,]^t$ and $W = [\, N \quad dE_n \,]$. By the straightforward calculation with noting that dE_m and dE_n are diagonal, we obtain the following relations:

$$\det(\Delta_I T) = \pm \det(\Delta_{\tau(I)} W^t) d^{m-n}.$$

Thus $t_{1I} = \pm w_{1\tau(I)} d^{m-n}$ for all $I \in \mathcal{I}$. It follows that the ideals $\sum_{I \in \mathcal{I}} (t_{1I})$ and $\sum_{J \in \mathcal{J}} (w_{1J})$ are isomorphic to each other. □

NOTE 4.3. The reduced minors are derived from t_I's and w_J's in Definition3.5. Thus t_I's and w_J's can be considered more primitive than the reduced minors. Nevertheless since we will present in Theorem 5.2 that t_I's and w_J's (or the ideals $\mathfrak{t}$ and $\mathfrak{w}$ generated by them) have the capability to state feedback stabilizability over commutative rings, we here consider that the full-size minor ideal $\mathfrak{t}$ (or the ideal $\mathfrak{w}$) is a generalization of the reduced minors.

5. Feedback Stabilizability in terms of Full-Size Minor Ideal. In this section, we present the necessary and sufficient condition of the feedback stabilizability over commutative rings in terms of the full-size minor ideal.

Let us consider the case where the set $\mathcal{A}$ of the stable causal transfer functions is not a unique factorization domain. Then it is not sufficient to use the family of reduced minors in order to state the feedback stabilizability. To see this, let us consider the result given by Anantharam in [1][1].

EXAMPLE 5.1. In [1], Anantharam considered the case where $\mathbb{Z}[\sqrt{5}\mathrm{i}]$ ($\simeq \mathbb{Z}[x]/(x^2+5)$) is the set of stable causal transfer functions, where $\mathbb{Z}$ is the ring of integers and i the imaginary unit; that is, $\mathcal{A} = \mathbb{Z}[\sqrt{5}\mathrm{i}]$. The set of all possible transfer functions is given as the field of fractions of $\mathcal{A}$; that is, $\mathcal{F} = \mathbb{Q}(\sqrt{5}\mathrm{i})$. In this case we have multiple factorizations $2 \cdot 3 = (1 + \sqrt{5}\mathrm{i})(1 - \sqrt{5}\mathrm{i})$ over $\mathcal{A}$, so that $\mathcal{A}$ is not a unique factorization domain. Anantharam in [1] considered the single-input single-output case and showed that the plant $p = (1 + \sqrt{5}\mathrm{i})/2$ does not have its coprime factorization over $\mathcal{A}$ but is stabilizable.

Now let $T = [\, 1 + \sqrt{5}\mathrm{i} \quad 2 \,]^t$. Since the plant p is of the single-input single-output ($m = n = 1$), we have $\mathcal{I} = \{\{1\}, \{2\}\}$. Thus let $I_1 = \{1\}$ and $I_2 = \{2\}$ so that $\mathcal{I} = \{I_1, I_2\}$. The full-size minors of the matrix T are $t_{I_1} = \det(\Delta_{I_1} T) = 1 + \sqrt{5}\mathrm{i}$ and $t_{I_2} = \det(\Delta_{I_2} T) = 2$. If Theorem 3.6 (or equivalently Corollary 3.7)

[1]The author wishes to thank to Dr. A. Quadrat (Centre d'Enseignement et de Recherche en Mathématiques, Informatique et Calcul Scientifique, ENPC, France) who introduced him to the paper of Anantharam[1].

could be applied even over a general commutative ring, the ideal (t_{I_1}, t_{I_2}) should be principal. However, the ideal (t_{I_1}, t_{I_2}) is not principal since p does not have its coprime factorization. ∎

In order to involve even such an example as a system over commutative ring, we extend Theorem 3.6. Since we cannot use the reduced minors to state the feedback stabilizability in general, we alternatively employ the full-size minor ideal $\mathfrak{t}$ rather than the reduced minors. The extension is the first main result of this paper and stated as follows.

THEOREM 5.2. *Let P be a causal plant of $\mathcal{P}^{n\times m}$. Then the plant P is stabilizable if and only if the full-size minor ideal $\mathfrak{t}$ of the plant P is projective. Further when $\mathfrak{t}$ is projective, it is of rank 1.*

By virtue of Lemma 4.2, the above theorem can be also stated with the ideal $\mathfrak{w}$ instead of the full-size minor ideal $\mathfrak{t}$.

In the case where $\mathcal{A}$ is a unique factorization domain, as in Theorem 3.6, the condition of feedback stabilizability is that the full-size minor ideal is free. On the other hand, in Theorem 5.2, the condition is that the ideal is projective. They are equivalent to each other in the case where $\mathcal{A}$ is a unique factorization domain as follows.

PROPOSITION 5.3. *Let $\mathcal{R}$ be a unique factorization domain. Then the ideal generated by finite elements of $\mathcal{R}$ is projective if and only if it is free.*
This proof will be given after finishing the proof of Theorem 5.2.

Now that we have presented the statement of Theorem 5.2, the main objective of the remainder of this section is to carry out the proof of Theorem 5.2. To do so, we prepare two main intermediate results. The first one is about the existence of right-/left-coprime factorizations of stabilizable plants over local rings, which will be presented in Subsection 5.1. The other is about the local-global principle of the feedback stabilizability, which will be presented in Subsection 5.2. Then we will prove Theorem 5.2. After the proof of Theorem 5.2 we will prove Proposition 5.3. Before finishing this section, we will present the relationship among the full-size minor ideals of P, C, and $H(P, C)$.

5.1. Right-/Left-Coprime Factorizations over Local Rings. The following is the first intermediate result of Theorem 5.2 about the existence of right-/left-coprime factorizations of stabilizable plants over local rings.

PROPOSITION 5.4. *Let P be a plant in $\mathcal{F}^{n\times m}$. Suppose that $\mathcal{R}$ is $\mathcal{A}_{\mathfrak{p}}$ with a prime ideal $\mathfrak{p}$ in $\mathrm{Spec}(\mathcal{A})$. Then the following statements are equivalent:*

(i) *The plant P is $\mathcal{R}$-stabilizable.*

(ii) *There exists a right-coprime factorization over $\mathcal{R}$ of P.*

(iii) *There exists a left-coprime factorization over $\mathcal{R}$ of P.*

The proof of this proposition will be presented after giving several its intermediate results.

We here recall the notion of Hermite used in [20][2], which can characterize the existence of both right-/left-coprime factorizations of transfer matrices.

DEFINITION 5.5. ([20, p.345]) *Let $\mathcal{R}$ be a commutative ring and A a matrix over $\mathcal{R}$ of size $x \times y$ with $x < y$. Then we say that the matrix A can be* complemented *if there exists a unimodular matrix in $(\mathcal{R})_y$ containing the matrix A as a submatrix. A row $[\, a_1 \quad \cdots \quad a_y \,] \in \mathcal{R}^{1\times y}$ is said to be a* unimodular row *if $a_1, \ldots, a_y$ together generate $\mathcal{R}$. A commutative ring $\mathcal{R}$ is said to be* Hermite *if every unimodular row can be complemented.*

The following result was given in [20] provided that $\mathcal{R}$ is an integral domain.

THEOREM 5.6. (cf. Theorem 8.1.66 of [20]) *Let $\mathcal{R}$ be a commutative ring. The following three statements are equivalent:*

(i) *The commutative ring $\mathcal{R}$ is Hermite.*

(ii) *If a matrix over $\mathcal{F}(\mathcal{R})$ has a right-coprime factorization over $\mathcal{R}$, it has also a left-coprime factorization over $\mathcal{R}$.*

(iii) *If a matrix over $\mathcal{F}(\mathcal{R})$ has a left-coprime factorization over $\mathcal{R}$, it has also a right-coprime factorization over $\mathcal{R}$.*

The "integral domain" version of this theorem was given as Theorem 8.1.66 of [20]. Even in the case of commutative rings, the proof is similar with that of Theorem 8.1.66 of [20] and so is omitted.

The following result is the intermediate result of Proposition 5.4, which makes the result above applicable to the proof of the proposition.

LEMMA 5.7. *Any local ring is Hermite.*

Proof. Suppose that $\mathcal{R}$ is a local ring and $[\, a_1, \ldots, a_y \,] \in \mathcal{R}^{1\times y}$ is a unimodular row. Thus there exist $b_1, \ldots, b_y \in \mathcal{R}$ such that

$$a_1 b_1 + \cdots + a_y b_y = 1. \tag{5.1}$$

Since $\mathcal{R}$ is local, the set of all nonunits is an ideal. From (5.1), there exists an i with $1 \le i \le y$ such that a_i is a unit. We assume without loss of generality that a_1

[2]It should be noted that this definition of "Hermite" is different from [6, 13].

is a unit. If $y = 1$, then a_1 is a unit, which can be considered as a unimodular matrix of $(\mathcal{R})_1$. In the following we consider the case $y > 1$. Then we can construct a unimodular matrix $U = (u_{ij}) \in (\mathcal{R})_y$:

$$u_{ij} = \begin{cases} a_j & \text{if } i = 1, \\ a_1^{-1} & \text{if } i = j = 2, \\ 1 & \text{if } i = j > 2, \\ 0 & \text{otherwise.} \end{cases}$$

This U contains the row $[\, a_1, \ldots, a_y \,]$ as a submatrix and hence every unimodular row can be complemented. Therefore $\mathcal{R}$ is Hermite. □

We prepare one more result which will help us present a nonsingular denominator matrix of a stabilizing controller

LEMMA 5.8. *Let $\mathcal{R}$ be a commutative ring and p a prime ideal of $\mathcal{R}$. Suppose that there exist matrices A, B, C_1, C_2 over $\mathcal{R}$ such that the determinant of the following square matrix is in $\mathcal{R}\backslash\mathrm{p}$:*

$$\begin{bmatrix} A & C_1 \\ B & C_2 \end{bmatrix}, \tag{5.2}$$

where the matrix A is square and the matrices A and B have same number of columns. Then there exists a matrix R over $\mathcal{R}$ such that the determinant of the matrix $A + RB$ is in $\mathcal{R}\backslash\mathrm{p}$.

Before starting the proof, it is worth reviewing some easy facts about a prime ideal.

Remark 5.9. Suppose that p is a prime ideal of $\mathcal{R}$. (i) If a is in $\mathcal{R}\backslash\mathrm{p}$ and expressed as $a = b + c$ with $b, c \in \mathcal{R}$, then at least one of b and c is in $\mathcal{R}\backslash\mathrm{p}$. (ii) If a is in $\mathcal{R}\backslash\mathrm{p}$ and b in p, then the sum $a + b$ is in $\mathcal{R}\backslash\mathrm{p}$. (iii) Every factor in $\mathcal{R}$ of an element of $\mathcal{R}\backslash\mathrm{p}$ belongs to $\mathcal{R}\backslash\mathrm{p}$ (that is, if $a, b \in \mathcal{R}$ and $ab \in \mathcal{R}\backslash\mathrm{p}$, then $a, b \in \mathcal{R}\backslash\mathrm{p}$).

Proof of Lemma 5.8. This proof mainly follows that of Lemma 4.4.21 of [20].

If $\det(A)$ is in $\mathcal{R}\backslash\mathrm{p}$, then we can select the zero matrix as R. Thus we assume in the following that $\det(A)$ is in p.

Since the determinant of (5.2) is in $\mathcal{R}\backslash\mathrm{p}$, there exists a full-size minor of $[\, A^t \quad B^t \,]^t$ in $\mathcal{R}\backslash\mathrm{p}$ by Laplace's expansion of (5.2) and by Remark 5.9(i,iii). Let a be such a full-size minor of $[\, A^t \quad B^t \,]^t$ having as few rows from B as possible.

We here construct a matrix R such that $\det(A + RB) = \pm a + z$ with a $z \in \mathrm{p}$. Since $\det(A) \in \mathrm{p}$, the full-size minor a must contain at least one row of B from the matrix $[\, A^t \quad B^t \,]^t$. Suppose that a is obtained by excluding the rows

$i_1,\dots,i_k$ of A and including the rows $j_1,\dots,j_k$ of B. Now define $R=(r_{ij})$ by $r_{i_1j_1}=\cdots=r_{i_kj_k}=1$ and $r_{ij}=0$ for all other i, j. Observe that $\det(A+RB)$ is expanded in terms of full-size minors of the matrices $[\,E\quad R\,]$ and $[\,A^t\quad B^t\,]^t$ from the factorization $A+RB=[\,E\quad R\,][\,A^t\quad B^t\,]^t$ by the Binet-Cauchy formula. Every minor of $[\,E\quad R\,]$ containing more than k columns of R is zero. By the method of choosing the rows from $[\,A^t\quad B^t\,]^t$ for the full-size minor a, every full-size minor of $[\,A^t\quad B^t\,]^t$ having less than k rows of B is in p. There is only one nonzero minor of $[\,E\quad R\,]$ containing exactly k columns of R, which is obtained by excluding the columns $i_1,\dots,i_k$ of the identity matrix E and including the columns $j_1,\dots,j_k$ of R; it is equal to ±1. From the Binet-Cauchy formula the corresponding minor of $[\,A^t\quad B^t\,]^t$ is a. As a result, $\det(A+RB)$ is given as a sum of $\pm a$ and elements in p. By Remark 5.9(ii), the sum is in $\mathcal{R}\backslash\mathrm{p}$ and so is $\det(A+RB)$. ∎

Now that we have the result above, we can prove Proposition 5.4.

Proof of Proposition 5.4. Since $\mathcal{R}$ is local, (ii) and (iii) are equivalent by Theorem 5.6 and Lemma 5.7. Thus we only prove (i)→(ii) and *vice versa*.

(i)→(ii). Suppose that P is $\mathcal{R}$-stabilizable. Then the $\mathcal{R}$-module $\mathcal{T}_{P,\mathcal{R}}$ is projective by Proposition 2.1 of [15]. Further it is free by Corollary 3.5 of [7, Ch.IV]. Let N and D be matrices over $\mathcal{R}$ with $P=ND^{-1}\in\mathcal{F}(\mathcal{R})$. Then the $\mathcal{R}$-module $M_r([\,N^t\quad D^t\,]^t)$ is free of rank m since D is nonsingular over $\mathcal{R}$. Let $v_1,v_2,\dots,v_m\in\mathcal{R}^m$ be a basis of the module $M_r([\,N^t\quad D^t\,]^t)$ and V the matrix of $(\mathcal{R})_m$ whose rows are $v_1,v_2,\dots,v_m$. Then, the matrix $[\,N^t\quad D^t\,]^t$ can be written in the form $[\,N^t\quad D^t\,]^t=[\,N_0^t\quad D_0^t\,]^t\,V$ by uniquely choosing the matrices N_0 in $\mathcal{R}^{n\times m}$ and D_0 in $(\mathcal{R})_m$. Because of $\det(D)=\det(D_0V)$, $\det(D_0)$ is a nonzerodivisor. It follows that $P=N_0D_0^{-1}$ over $\mathcal{F}(\mathcal{R})$. In the following we show that the matrices N_0 and D_0 are right-coprime over $\mathcal{R}$. Since $v_1,\dots,v_m$ belong to $M_r([\,N^t\quad D^t\,]^t)$, there exist matrices $\widetilde{Y}$ in $\mathcal{R}^{m\times n}$ and $\widetilde{X}$ in $(\mathcal{R})_m$ such that $V=[\,\widetilde{Y}\quad\widetilde{X}\,][\,N^t\quad D^t\,]^t$. So we have $V=(\widetilde{Y}N_0+\widetilde{X}D_0)V$. Since V is nonsingular, we obtain $\widetilde{Y}N_0+\widetilde{X}D_0=E_m$ over $\mathcal{R}$. Thus (N_0,D_0) is a right-coprime factorization over $\mathcal{R}$ of P.

(ii)→(i). Suppose that there exists a right-coprime factorization over $\mathcal{R}$ of the plant P; that is, there exist the matrices N, D, $\widetilde{Y}$, $\widetilde{X}$ over $\mathcal{R}$ with $\widetilde{Y}N+\widetilde{X}D=E_m$ and $P=ND^{-1}$. If $\det(\widetilde{X})$ is a nonzerodivisor of $\mathcal{R}$, it is obvious that $\widetilde{X}^{-1}\widetilde{Y}$ is an $\mathcal{R}$-stabilizing controller. Thus in the following we suppose that $\det(\widetilde{X})$ is a zerodivisor of $\mathcal{R}$.

By the equivalence between (ii) and (iii), there also exists a left-coprime factorization over $\mathcal{R}$ of P; that is, there exist the matrices $\widetilde{N}$, $\widetilde{D}$, Y, X over $\mathcal{R}$ with

$\widetilde{N}Y + \widetilde{D}X = E_n$ and $P = \widetilde{D}^{-1}\widetilde{N}$. Thus we have the following matrix equation:

$$\begin{bmatrix} \widetilde{X} & \widetilde{Y} \\ \widetilde{N} & -\widetilde{D} \end{bmatrix} \begin{bmatrix} D & Y \\ N & -X \end{bmatrix} = \begin{bmatrix} E_m & \widetilde{X}Y - \widetilde{Y}X \\ O & E_n \end{bmatrix}. \tag{5.3}$$

Observe that the determinant of the right-hand side of the matrix equation above is in $\mathcal{R}\backslash\mathcal{Z}_{\mathrm{p}}$, where $\mathcal{Z}_{\mathrm{p}}$ denotes the localization of the prime ideal $\mathcal{Z}$ at p (Note that $\mathcal{Z}_{\mathrm{p}}$ is also a prime ideal of $\mathcal{R}$). Hence the determinant of the first matrix in (5.3) is in $\mathcal{R}\backslash\mathcal{Z}_{\mathrm{p}}$ again. Applying Lemma 5.8 to the first matrix, we have a matrix R over $\mathcal{R}$ such that the determinant of the matrix $\widetilde{X} + R\widetilde{N}$ is in $\mathcal{R}\backslash\mathcal{Z}_{\mathrm{p}}$. Now $(\widetilde{X} + R\widetilde{N})^{-1}(\widetilde{Y} - R\widetilde{D})$ is an $\mathcal{R}$-stabilizing controller. $\square$

5.2. Local-Global Principle in Stabilizability. Next we present the local-global principle below about the feedback stabilizability as the second intermediate result of this section.

PROPOSITION 5.10. *Suppose that the plant P is causal. Then the following statements are equivalent:*

(i) *P is stabilizable.*

(ii) *P is $\mathcal{A}_{\mathrm{p}}$-stabilizable for each prime ideal p in $\mathrm{Spec}(\mathcal{A})$.*

(iii) *P is $\mathcal{A}_{\mathrm{m}}$-stabilizable for each maximal ideal m in $\mathrm{Max}(\mathcal{A})$.*

(iv) *For every prime ideal p in $\mathrm{Spec}(\mathcal{A})$, P has either its right- or left-coprime factorization over $\mathcal{A}_{\mathrm{p}}$.*

(v) *For every maximal ideal m in $\mathrm{Max}(\mathcal{A})$, P has either its right- or left-coprime factorization over $\mathcal{A}_{\mathrm{m}}$.*

Further, if P is stabilizable, then there exists a causal stabilizing controller of P.

Note here that by virtue of Proposition 5.4, if (iv) holds (if (v) holds), then the plant P has both right-/left-coprime factorizations over $\mathcal{A}_{\mathrm{p}}$ (over $\mathcal{A}_{\mathrm{m}}$).

We consider that this is a generalization of Proposition 2 of [18] in which the strict causality of the plant is assumed (see [19] for the definition of the strict causality). On the other hand, we assume only that the plant is causal.

Now we begin to prove Proposition 5.10.

Proof of Proposition 5.10. Since the following implications are obvious:

$$\begin{tikzcd}
 & & \text{(iii)} \arrow[dr, leftrightarrow] & \\
\text{(i)} \arrow[r] & \text{(ii)} \arrow[ur] \arrow[dr, leftrightarrow] & & \text{(v)} \\
 & & \text{(iv)} \arrow[ur] &
\end{tikzcd}$$

by virtue of Proposition 5.4, we only show that (v) implies (i).

Suppose that (v) holds. Let N, D, $\widetilde{N}$, and $\widetilde{D}$ be matrices over $\mathcal{A}$ with $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ such that D and $\widetilde{D}$ are $\mathcal{Z}$-nonsingular (recall that P is causal). By Proposition 5.4, P has both right-/left-coprime factorizations over $\mathcal{A}_{\mathrm{m}}$ with $\mathrm{m} \in \mathrm{Max}(\mathcal{A})$. As in the proof of Proposition 5.4, for each m in $\mathrm{Max}(\mathcal{A})$, there exist matrices Y_{m}, X_{m}, $\widetilde{Y}_{\mathrm{m}}$, $\widetilde{X}_{\mathrm{m}}$, N_{m}, D_{m}, $\widetilde{N}_{\mathrm{m}}$, $\widetilde{D}_{\mathrm{m}}$, V_{m}, and W_{m} over $\mathcal{A}_{\mathrm{m}}$ such that

$$\begin{bmatrix} N \\ D \end{bmatrix} = \begin{bmatrix} N_{\mathrm{m}} \\ D_{\mathrm{m}} \end{bmatrix} V_{\mathrm{m}}, \quad [\, \widetilde{N} \quad \widetilde{D} \,] = W_{\mathrm{m}} [\, \widetilde{N}_{\mathrm{m}} \quad \widetilde{D}_{\mathrm{m}} \,], \tag{5.4}$$

$$\widetilde{Y}_{\mathrm{m}} N_{\mathrm{m}} + \widetilde{X}_{\mathrm{m}} D_{\mathrm{m}} = E_m, \quad \widetilde{N}_{\mathrm{m}} Y_{\mathrm{m}} + \widetilde{D}_{\mathrm{m}} X_{\mathrm{m}} = E_n \tag{5.5}$$

hold over $\mathcal{A}_{\mathrm{m}}$. For each $\mathrm{m} \in \mathrm{Max}(\mathcal{A})$ let q_{m} be an arbitrary but fixed element of $\mathcal{A} \backslash \mathrm{m}$ such that the six matrices $q_{\mathrm{m}} N_{\mathrm{m}} \widetilde{Y}_{\mathrm{m}}$, $q_{\mathrm{m}} N_{\mathrm{m}} \widetilde{X}_{\mathrm{m}}$, $q_{\mathrm{m}} D_{\mathrm{m}} \widetilde{Y}_{\mathrm{m}}$, $q_{\mathrm{m}} D_{\mathrm{m}} \widetilde{X}_{\mathrm{m}}$, $q_{\mathrm{m}} \widetilde{D}_{\mathrm{m}}$, and $q_{\mathrm{m}} \widetilde{N}_{\mathrm{m}}$ are over $\mathcal{A}$.

For a subset $\mathcal{B}$ of $\mathcal{A}$, denote by $\Gamma(\mathcal{B})$ the set of all maximal ideals m of $\mathcal{A}$ with $\mathcal{B} \not\subset \mathrm{m}$, that is, $\Gamma(\mathcal{B}) = \{\mathrm{m} \in \mathrm{Max}(\mathcal{A}) \,|\, \mathcal{B} \not\subset \mathrm{m}\}$. Since $q_{\mathrm{m}} \in \mathcal{A} \backslash \mathrm{m}$, we have $\mathrm{m} \in \Gamma(\mathcal{A} q_{\mathrm{m}})$. Thus $\mathrm{Max}(\mathcal{A}) = \bigcup_{\mathrm{m} \in \mathrm{Max}(\mathcal{A})} \Gamma(\mathcal{A} q_{\mathrm{m}})$. Recall that $\mathrm{Max}(\mathcal{A})$ is compact (see Theorem IV.1 of [13]). Hence there are a finite number of $\mathrm{m}_1, \ldots, \mathrm{m}_t$ of maximal ideals such that $\mathrm{Max}(\mathcal{A}) = \bigcup_{i=1}^{t} \Gamma(\mathcal{A} q_{\mathrm{m}_i})$. It follows that $\mathrm{Max}(\mathcal{A}) = \Gamma(\sum_{i=1}^{t} \mathcal{A} q_{\mathrm{m}_i})$ and, consequently, $\mathcal{A} = \sum_{i=1}^{t} \mathcal{A} q_{\mathrm{m}_i}$. Therefore there exist $r_1, \ldots, r_t$ in $\mathcal{A}$ with $1 = r_1 q_{\mathrm{m}_1} + \cdots + r_t q_{\mathrm{m}_t}$.

Next we want to consider that at least one of $\mathrm{m}_1, \ldots, \mathrm{m}_t$ contains $\mathcal{Z}$. In the case where every m_i in $\mathrm{m}_1, \ldots, \mathrm{m}_t$ does not contain $\mathcal{Z}$, we reconstruct t, r_i's, and q_{m_i}'s as follows. We first pick an $\mathrm{m}_{t+1} \in \mathrm{Max}(\mathcal{A})$ with $\mathrm{m}_{t+1} \supset \mathcal{Z}$. Then we let r_i be $(1 - q_{\mathrm{m}_{t+1}}) r_i$ for $1 \le i \le t$ and $r_{t+1} = 1$. We now let $t := t + 1$. Then we have again $1 = r_1 q_{\mathrm{m}_1} + \cdots + r_t q_{\mathrm{m}_t}$ and, in this case, $\mathrm{m}_t \supset \mathcal{Z}$. Hence we can assume without loss of generality that at least one of $\mathrm{m}_1, \ldots, \mathrm{m}_t$, say m_1, contains $\mathcal{Z}$.

Observe then that the following equality holds:

$$\begin{aligned} 1 = (r_1 q_{\mathrm{m}_1} + r_1 - 1) q_{\mathrm{m}_1} &+ (r_2 q_{\mathrm{m}_1} + r_2) q_{\mathrm{m}_2} \\ &+ \cdots + (r_t q_{\mathrm{m}_1} + r_t) q_{\mathrm{m}_t}. \end{aligned} \tag{5.6}$$

At least one of $r_1 q_{\mathrm{m}_1} + r_1 - 1$ and r_1 must be in $\mathcal{A} \backslash \mathcal{Z}$. Thus in the case $r_1 \in \mathcal{Z}$, we can reassign r_i's as in (5.6), so that r_1 is in $\mathcal{A} \backslash \mathcal{Z}$. Therefore we can assume without loss of generality that $r_1 q_{\mathrm{m}_1} \in \mathcal{A} \backslash \mathcal{Z}$.

Consider here the following matrix

$$\begin{bmatrix} E_n - \sum_{i=1}^{t} r_i q_{\mathrm{m}_i} N_{\mathrm{m}_i} \widetilde{Y}_{\mathrm{m}_i} & -\sum_{i=1}^{t} r_i q_{\mathrm{m}_i} N_{\mathrm{m}_i} \widetilde{X}_{\mathrm{m}_i} \\ \sum_{i=1}^{t} r_i q_{\mathrm{m}_i} D_{\mathrm{m}_i} \widetilde{Y}_{\mathrm{m}_i} & \sum_{i=1}^{t} r_i q_{\mathrm{m}_i} D_{\mathrm{m}_i} \widetilde{X}_{\mathrm{m}_i} \end{bmatrix}, \tag{5.7}$$

which is over $\mathcal{A}$. For short we partition (5.7) as

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}.$$

In the case where H_{22} is $\mathcal{Z}$-nonsingular, letting $C = H_{22}^{-1}H_{21} \in \mathcal{P}^{m\times n}$ we can check that $H(P,C)$ is equal to (5.7), which implies that P is stabilized by C. Hence in the rest of this proof we show that if H_{22} is $\mathcal{Z}$-singular, then H_{22} can be made $\mathcal{Z}$-nonsingular by reassigning $\widetilde{X}_{\mathrm{m}_i}$ and $\widetilde{Y}_{\mathrm{m}_i}$ for an i.

First we show the $\mathcal{Z}$-nonsingularity of the matrices $r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1}$ and $r_1q_{\mathrm{m}_1}\widetilde{D}_{\mathrm{m}_1}$. Since $r_1q_{\mathrm{m}_1} \in \mathcal{A}\backslash\mathcal{Z}$, we have $\det(r_1q_{\mathrm{m}_1}D) \in \mathcal{A}\backslash\mathcal{Z}$. From the first matrix equation of (5.4), we have $\det(r_1q_{\mathrm{m}_1}D) = \det(r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1})\det(V_{\mathrm{m}_1})$. Hence $r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1}$ is $\mathcal{Z}$-nonsingular. Analogously, from the second matrix equation of (5.4), $r_1q_{\mathrm{m}_1}\widetilde{D}_{\mathrm{m}_1}$ is $\mathcal{Z}$-nonsingular.

Next consider the following matrix equation over $\mathcal{A}$:

$$\begin{bmatrix} \sum_{i=1}^{t} r_iq_{\mathrm{m}_i}D_{\mathrm{m}_i}\widetilde{X}_{\mathrm{m}_i} & \sum_{i=1}^{t} r_iq_{\mathrm{m}_i}D_{\mathrm{m}_i}\widetilde{Y}_{\mathrm{m}_i} \\ -r_1q_{\mathrm{m}_1}\det(r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1})\widetilde{N}_{\mathrm{m}_1} & r_1q_{\mathrm{m}_1}\det(r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1})\widetilde{D}_{\mathrm{m}_1} \end{bmatrix} \times \begin{bmatrix} D & O \\ N & E_n \end{bmatrix} = \begin{bmatrix} D & \sum_{i=1}^{t} r_iq_{\mathrm{m}_i}D_{\mathrm{m}_i}\widetilde{Y}_{\mathrm{m}_i} \\ O & r_1q_{\mathrm{m}_1}\det(r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1})\widetilde{D}_{\mathrm{m}_1} \end{bmatrix}. \tag{5.8}$$

Since the matrices D, $r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1}$, and $r_1q_{\mathrm{m}_1}\widetilde{D}_{\mathrm{m}_1}$ are $\mathcal{Z}$-nonsingular, so is the right-hand side of (5.8). Thus the first matrix of (5.8) is also $\mathcal{Z}$-nonsingular. By Lemma 5.8 and the first matrix of (5.8), there exists a matrix R'_{m_1} of $\mathcal{A}^{m\times n}$ such that the following matrix is $\mathcal{Z}$-nonsingular:

$$\sum_{i=1}^{t} r_iq_{\mathrm{m}_i}D_{\mathrm{m}_i}\widetilde{X}_{\mathrm{m}_i} - r_1q_{\mathrm{m}_1}\det(r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1})R'_{\mathrm{m}_1}\widetilde{N}_{\mathrm{m}_1}.$$

Now let R_{m_1} be $r_1q_{\mathrm{m}_1}\operatorname{adj}(r_1q_{\mathrm{m}_1}D_{\mathrm{m}_1})R'_{\mathrm{m}_1}$. Further we let $\widetilde{X}_{\mathrm{m}_1}$ be the matrix $\widetilde{X}_{\mathrm{m}_1} - R_{\mathrm{m}_1}\widetilde{N}_{\mathrm{m}_1}$ and $\widetilde{Y}_{\mathrm{m}_1}$ the matrix $\widetilde{Y}_{\mathrm{m}_1} + R_{\mathrm{m}_1}\widetilde{D}_{\mathrm{m}_1}$, which are consistent with (5.5). Thus we can now consider without loss of generality that the matrix $\sum_{i=1}^{t} r_iq_{\mathrm{m}_i}D_{\mathrm{m}_i}\widetilde{X}_{\mathrm{m}_i}$ is $\mathcal{Z}$-nonsingular and so is H_{22}. □

5.3. Proof of Theorem 5.2. Before proving Theorem 5.2, we should prepare a small result.

LEMMA 5.11. *Let* $a \in \mathcal{A}$ *and* $\mathrm{p} \in \operatorname{Spec}(\mathcal{A})$. *Then* $(a)_{\mathrm{p}}$ *and* $(a/1)$ *are isomorphic to each other as* $\mathcal{A}_{\mathrm{p}}$*-modules, where* $(a)_{\mathrm{p}}$ *denotes the localization,*

at p, *of the principal ideal generated by* $a \in \mathcal{A}$ *and* $(a/1)$ *the principal ideal generated by* $a/1 \in \mathcal{A}_{\mathrm{p}}$.

The proof of the lemma is elementary and is omitted.

Now we start to prove the first result of this paper.

Proof of Theorem 5.2. We show first the "Only If" part and then the "If" part. (Only If). Suppose that P is stabilizable. Then by Proposition 5.10, for every prime ideal p in $\mathrm{Spec}(\mathcal{A})$, P is $\mathcal{A}_{\mathrm{p}}$-stabilizable. By Proposition 5.4, P has both its right-/left-coprime factorizations over $\mathcal{A}_{\mathrm{p}}$. Suppose that $\widetilde{Y}_{\mathrm{p}} N_{\mathrm{p}} + \widetilde{X}_{\mathrm{p}} D_{\mathrm{p}} = E_m$ holds over $\mathcal{A}_{\mathrm{p}}$ with $P = N_{\mathrm{p}} D_{\mathrm{p}}^{-1}$, where the matrices N_{p}, D_{p}, $\widetilde{Y}_{\mathrm{p}}$, and $\widetilde{X}_{\mathrm{p}}$ are over $\mathcal{A}_{\mathrm{p}}$. Then let $T_{\mathrm{p}} = [\, N_{\mathrm{p}}^t \quad D_{\mathrm{p}}^t \,]^t$. By Binet-Cauchy formula we have $\sum_{I \in \mathcal{I}} (\det(\Delta_I T_{\mathrm{p}})) = \mathcal{A}_{\mathrm{p}}$. Thus by virtue of Lemmas 4.2 and 5.11, the ideal $\mathfrak{t}_{\mathrm{p}}$ is free (recall that $\mathfrak{t}_{\mathrm{p}}$ denotes the localization of the full-size minor ideal $\mathfrak{t}$ at p), which is also finitely generated. This holds for every prime ideal p. From Theorem IV.32 of [13], the full-size minor ideal $\mathfrak{t}$ is projective.

(If). Suppose that the full-size minor ideal $\mathfrak{t}$ is projective. Let p be a prime ideal in $\mathrm{Spec}(\mathcal{A})$. Then $\mathfrak{t}_{\mathrm{p}}$ is free by Theorem IV.32 of [13] again. Thus there exist g, a_I, and r_I in $\mathcal{A}_{\mathrm{p}}$ with $g = \sum_{I \in \mathcal{I}} r_I t_I$ and $t_I = a_I g$ for every $I \in \mathcal{I}$. Since $g = \sum_{I \in \mathcal{I}} r_I a_I g$ and g is a nonzerodivisor, we have $\sum_{I \in \mathcal{I}} r_I a_I = 1$. Recall here that $\mathcal{A}_{\mathrm{p}}$ is local. Hence the set of all nonunits in $\mathcal{A}_{\mathrm{p}}$ is an ideal. Thus there exists $I_0 \in \mathcal{I}$ such that $r_{I_0} a_{I_0}$ is a unit of $\mathcal{A}_{\mathrm{p}}$. This implies that a_{I_0} is a unit of $\mathcal{A}_{\mathrm{p}}$ and further that every t_I has a factor t_{I_0} over $\mathcal{A}_{\mathrm{p}}$ (that is, t_{I_0} and g are associate). Now let $T' = T \,\mathrm{adj}(\Delta_{I_0} T)$ and $t'_I = \det(\Delta_I T')$ for every $I \in \mathcal{I}$. Then $t'_I = t_I \det(\mathrm{adj}(\Delta_{I_0} T))$ and $\Delta_{I_0} T' = t_{I_0} E_m$ hold. Since $\det(\mathrm{adj}(\Delta_{I_0} T)) = t_{I_0}^{m-1}$, every t'_I has a common factor $t_{I_0}^m$.

Suppose that i is an integer with $i \notin I_0$ and $1 \leq i \leq m+n$. Suppose further that $i_{01}, \ldots, i_{0m}$ are elements in I_0 with ascending order. Now let $I = \{i, i_{01}, i_{02}, \ldots, i_{0\,k-1}, i_{0\,k+1}, \ldots, i_{0m}\}$. Then t_I is expressed as $\pm t_{ik} t_{I_0}^{m-1}$ where t_{ik} is the (i,k)-entry of the matrix T'. Since t'_I has a factor $t_{I_0}^m$, t_{ik} has a factor t_{I_0}. This fact holds for all i between $1 \leq i \leq m+n$ but $i \notin I_0$. As a result, t_{I_0} is a common factor of all entries of T'.

Let $T'' = T'/t_{I_0}$ over $\mathcal{A}_{\mathrm{p}}$. Since $\Delta_{I_0} T''$ is the identity matrix, the matrix Δ_{I_0} itself is a left inverse of T''. Let $\widetilde{Y}_{I_0}$ and $\widetilde{X}_{I_0}$ be matrices with $[\, \widetilde{Y}_{I_0} \quad \widetilde{X}_{I_0} \,] = \Delta_{I_0}$. Further we let N_{I_0} and D_{I_0} be matrices over $\mathcal{A}_{\mathrm{p}}$ with $T'' = [\, N_{I_0}^t \quad D_{I_0}^t \,]^t$. Then we obtain $\widetilde{Y}_{I_0} N_{I_0} + \widetilde{X}_{I_0} D_{I_0} = E_m$ over $\mathcal{A}_{\mathrm{p}}$, which is a right-coprime factorization over $\mathcal{A}_{\mathrm{p}}$ of the plant P. Therefore by Proposition 5.10, P is stabilizable. □

5.4. Proof of Proposition 5.3. Now we prove Proposition 5.3. We first prepare the following local-global principle on ideals.

LEMMA 5.12. *Let $\mathcal{R}$ be a commutative ring. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ be ideals of $\mathcal{R}$. Then the following statements are equivalent:*

(i) $\mathfrak{a}_1 + \cdots + \mathfrak{a}_k = \mathcal{R}$.

(ii) $\mathfrak{a}_{1\mathrm{p}} + \cdots + \mathfrak{a}_{k\mathrm{p}} = \mathcal{R}_\mathrm{p}$ *for all prime ideal* $\mathrm{p} \in \mathrm{Spec}(\mathcal{A})$.

(iii) $\mathfrak{a}_{1\mathrm{m}} + \cdots + \mathfrak{a}_{k\mathrm{m}} = \mathcal{R}_\mathrm{m}$ *for all maximal ideal* $\mathrm{m} \in \mathrm{Max}(\mathcal{A})$.

Proof. It is obvious that (i) implies (ii) and (ii) implies (iii). Hence we only show that (iii) implies (i).
(iii)$\rightarrow$(i). Suppose that (iii) holds. Let m be a maximal ideal of $\mathcal{A}$. Since $\mathcal{R}_\mathrm{m}$ is local, the set of all nonunits in $\mathcal{R}_\mathrm{m}$ is an ideal. Hence there exists an i_m with $1 \leq i_\mathrm{m} \leq k$ such that $\mathfrak{a}_{i_\mathrm{m}\mathrm{m}} = \mathcal{R}_\mathrm{m}$. Thus there exists s_m in $\mathcal{R}\backslash\mathrm{m}$ such that $s_\mathrm{m} \in \mathfrak{a}_{i_\mathrm{m}}$.

Recalling the proof of Proposition 5.10, we have a finite number of $\mathrm{m}_1, \ldots, \mathrm{m}_t$ in $\mathrm{Max}(\mathcal{R})$ and $r_1, \ldots, r_t \in \mathcal{R}$ such that $1 = r_1 s_{\mathrm{m}_1} + \cdots + r_t s_{\mathrm{m}_t}$ over $\mathcal{R}$. For every $l = 1$ to t, $r_l s_{\mathrm{m}_l}$ is an element of $\mathfrak{a}_i$ with $i = i_{\mathrm{m}_l}$. Therefore we have (i). □

Proof of Proposition 5.3. Suppose that $\mathcal{R}$ is a unique factorization domain. Since the "If" part is obvious, we prove only the "Only If" part.
(Only If). Let $a_1, \ldots, a_k$ be in $\mathcal{R}$. Suppose that $(a_1, \ldots, a_k)$ is projective. If all $a_1, \ldots, a_k$ are zero, the proof is obvious. Thus in the following we suppose that at least one of $a_1, \ldots, a_k$ is nonzero. Since $\mathcal{R}$ is a unique factorization domain, there exists a nonzero greatest common factor of a_i's, denoted by g. Thus there exist b_i's in $\mathcal{A}$ with $b_i g = a_i$. Then $(b_1, \ldots, b_k)$ is projective again. For any prime ideal p in $\mathrm{Spec}(\mathcal{R})$, $(b_1, \ldots, b_k)_\mathrm{p}$ is free of rank 1. Since there is no nonunit common factor among b_i's over $\mathcal{R}$, $(b_1, \ldots, b_k)_\mathrm{p} = \mathcal{R}_\mathrm{p}$. By Lemma 5.12, $(b_1, \ldots, b_k) = \mathcal{R}$. Hence $(a_1, \ldots, a_k) = (g)$, which is free. □

5.5. Full-Size Minor Ideals of P, C, and $H(P,C)$. Now that we have obtained Theorem 5.2, we know that the projectivity of the full-size minor ideal of the plant connects with the feedback stabilizability of the plant. Since P, C, and $H(P,C)$ are transfer matrices over $\mathcal{F}$, we can define the full-size minor ideals of C and $H(P,C)$ analogously to that of P.

We present here the relationship among the full-size minor ideals of P, C, and $H(P,C)$.

PROPOSITION 5.13. *Let $\mathfrak{t}_P$, $\mathfrak{t}_C$, $\mathfrak{t}_{H(P,C)}$ be the full-size minor ideals of P, C, and $H(P,C)$, respectively. Then $\mathfrak{t}_{H(P,C)}$ is isomorphic (as an $\mathcal{A}$-module) to the ideal generated by $t_1 t_2$'s for all $t_1 \in \mathfrak{t}_P$ and all $t_2 \in \mathfrak{t}_C$.*

This proposition holds even if C is not a stabilizing controller of P. Before proving this proposition, we present a preliminary lemma.

LEMMA 5.14. *Let A and B are matrices over $\mathcal{R}$ such that $B = UA$, where U is a unimodular matrix over $\mathcal{R}$. Then the ideal generated by the full-size minors of A is equal to that of B.*

The proof of this lemma is straightforward and omitted.

Proof of Proposition 5.13. By virtue of Lemma 4.2, we suppose without loss of generality that N and N_c are matrices over $\mathcal{A}$ and d and d_c in $\mathcal{A}$ with $P = Nd^{-1}$ and $C = N_c d_c^{-1}$. Let A and B be the following matrices:

$$A = \begin{bmatrix} N_c & O \\ d_c E_n & O \\ O & N \\ O & dE_m \end{bmatrix}, \quad B = \begin{bmatrix} Q \\ S \end{bmatrix}, \text{ where}$$

$$Q = \begin{bmatrix} d_c E_n & N \\ -N_c & dE_m \end{bmatrix}, \quad S = \begin{bmatrix} d_c E_n & O \\ O & dE_m \end{bmatrix}.$$

Then we can see that there exists a unimodular matrix U with $B = UA$ and that $H(P,C) = SQ^{-1}$. Let $\mathfrak{a}$ be the ideal generated by the full-size minors of A and $\mathfrak{t}_{P,C}$ be the ideal generated by $t_1 t_2$'s for all $t_1 \in \mathfrak{t}_P$ and all $t_2 \in \mathfrak{t}_C$. Then by Lemma 5.14, $\mathfrak{t}_{H(P,C)}$ is isomorphic to $\mathfrak{a}$ as $\mathcal{A}$-modules. Also by Binet-Cauchy formula, $\mathfrak{a} \simeq \mathfrak{t}_{P,C}$. Hence we obtain $\mathfrak{t}_{H(P,C)} \simeq \mathfrak{t}_{P,C}$. □

6. Stabilizability in terms of Coprimeness of Quotient Ideals. In this section, we present one more necessary and sufficient condition of the feedback stabilizability which is given in terms of quotient ideals.

THEOREM 6.1. *Let P be a causal plant of $\mathcal{P}^{n\times m}$. Then the plant P is stabilizable if and only if the ideal*

$$\sum_{I\in\mathcal{I}} ((t_I) : \mathfrak{t}) \tag{6.1}$$

is equal to $\mathcal{A}$.

The ideal of (6.1) will be considered as another generalization of the reduced minors. This will be presented later as Proposition 6.5.

We note that the result above can be considered as a generalization of Theorem 2.1.1 in [17] given by Shankar and Sule as well as a generalization of Theorem 3.6. They considered the single-input single-output case. In Theorem 2.1.1 of [17], they stated the feedback stabilizability of the given plant in terms of the coprimeness of the ideal quotients as (6.1). As a result, Theorem 6.1 can be considered as a multi-input multi-output version of Theorem 2.1.1 of [17].

In order to prove Theorem 6.1, we prepare a relationship between projective modules and quotient ideals as follows.

THEOREM 6.2. *Let $\mathcal{R}$ be a commutative ring and $a_1, \ldots, a_k \in \mathcal{R}$. Then $(a_1, \ldots, a_k)$, that is, the ideal generated by $a_1, \ldots, a_k$ is projective if and only if the following equation holds:*

$$\sum_{i=1}^{k}((a_i) : (a_1, \ldots, a_k)) = \mathcal{R}. \tag{6.2}$$

Once we obtain Theorem 6.2, the proof of Theorem 6.1 is directly obtained from Theorems 5.2 and 6.2. Thus we will present only the proof of Theorem 6.2, which will be given after showing intermediate results (Lemmas 6.3 and 6.4).

LEMMA 6.3. *Let $\mathcal{R}$ be a commutative ring and $a_1, \ldots, a_k \in \mathcal{R}$. If $(a_1, \ldots, a_k)$ is free, then (6.2) holds.*

Proof. As in the proof of Proposition 5.3, if all $a_1, \ldots, a_k$ are zero, the proof is obvious. Thus in the following we assume that at least one of $a_1, \ldots, a_k$ is nonzero. Then there exist a nonzero g in $\mathcal{R}$ and b_i in $\mathcal{R}$ for $i = 1$ to k such that $(g) = (a_1, \ldots, a_k)$ and $a_i = b_i g$. Thus there exist $r_i \in \mathcal{R}$ for $i = 1$ to k with $g = r_1 a_1 + \cdots + r_k a_k$. If g was a zerodivisor, the principal ideal (g) could not be free. Hence g is a nonzerodivisor. Now we have

$$r_1 b_1 + \cdots + r_k b_k = 1. \tag{6.3}$$

Since $b_i(a_1, \ldots, a_k) \subset (a_i)$ for all i, we have $b_i \in ((a_i) : (a_1, \ldots, a_k))$. It follows from (6.3) that we now have (6.2). $\square$

LEMMA 6.4. *Let $\mathcal{R}$ be a commutative ring, $\mathfrak{a}, \mathfrak{b}$ ideals of $\mathcal{R}$, and p a prime ideal of $\mathcal{R}$. Denote by $(\mathfrak{a} : \mathfrak{b})_{\mathrm{p}}$ the localization of the quotient ideal $(\mathfrak{a} : \mathfrak{b})$ at p. Further let $(\mathfrak{a}_{\mathrm{p}} : \mathfrak{b}_{\mathrm{p}})$ be the quotient ideal of $\mathcal{R}_{\mathrm{p}}$, where $\mathfrak{a}_{\mathrm{p}}$ and $\mathfrak{b}_{\mathrm{p}}$ are localizations of ideals $\mathfrak{a}$ and $\mathfrak{b}$ at p, respectively. Then $(\mathfrak{a} : \mathfrak{b})_{\mathrm{p}} = (\mathfrak{a}_{\mathrm{p}} : \mathfrak{b}_{\mathrm{p}})$ holds.*

Now we are in a position to prove Theorem 6.2.

Proof of Theorem 6.2. By the same reason as in the proofs of Proposition 5.3 and Lemma 6.3, we assume that at least one of $a_1, \ldots, a_k$ is nonzero.

(If). Suppose that (6.2) holds. Then there exist $x_i \in ((a_i) : (a_1, \ldots, a_k))$ for $i = 1$ to k such that $1 = \sum_{i=1}^{k} x_i$. By appropriate changes of $a_1, \ldots, a_k$, we assume without loss of generality that all $x_1, \ldots, x_{k'}$ are nonzero with $1 \le k' \le k$ and all $x_{k'+1}, \ldots, x_k$ are zero subject to $k' < k$. Observe that for each i between 1 and k', $(a_1, \ldots, a_k)_{x_i} = (a_i)_{x_i}$ over $\mathcal{A}_{x_i}$, where $(a_1, \ldots, a_k)_{x_i}$ and $(a_i)_{x_i}$ denote the localizations of $(a_1, \ldots, a_k)$ and (a_i) at x_i, respectively. Hence for each i

between 1 and k', $(a_1, \dots, a_k)_{x_i}$ is free over $\mathcal{A}_{x_i}$. Therefore by Theorem IV.32 of [13], $(a_1, \dots, a_k)$ is projective as $\mathcal{R}$-module.
(Only If). Suppose that $(a_1, \dots, a_k)$ is projective. Then again by Theorem IV.32 of [13], for each p in $\mathrm{Spec}(\mathcal{R})$, $(a_1, \dots, a_k)_{\mathrm{p}}$ is free over $\mathcal{R}_{\mathrm{p}}$. By Lemma 6.3, we have

$$\sum_{i=1}^{k} ((a_i)_{\mathrm{p}} : (a_1, \dots, a_k)_{\mathrm{p}}) = \mathcal{R}_{\mathrm{p}} \tag{6.4}$$

for each p in $\mathrm{Spec}(\mathcal{R})$. Then (6.4) can be rewritten as follows by Lemma 6.4:

$$\sum_{i=1}^{k} ((a_i) : (a_1, \dots, a_k))_{\mathrm{p}} = \mathcal{R}_{\mathrm{p}}. \tag{6.5}$$

Since this holds for every p in $\mathrm{Spec}(\mathcal{A})$, applying Lemma 5.12 to (6.5) we obtain (6.2). $\square$

We now connect the reduced minors with the quotient ideal of (6.1) provided that $\mathcal{A}$ is a unique factorization domain.

PROPOSITION 6.5. *Suppose that $\mathcal{A}$ is a unique factorization domain. Let a_I denote the reduced minor of the matrix T with respect to $I \in \mathcal{I}$. Then $(a_I) = ((t_I) : \mathfrak{t})$ holds for every $I \in \mathcal{I}$.*

Proof. We first show (i) $(a_I) \subset ((t_I) : \mathfrak{t})$ and then (ii) the opposite inclusion.
(i). For every $I' \in \mathcal{I}$, $a_I t_{I'} = a_{I'} t_I$ holds, which implies that $a_I \in ((t_I) : (t_{I'}))$. Hence $a_I \in ((t_I) : \mathfrak{t})$.
(ii). Suppose that λ_I is an element of the quotient ideal $((t_I) : \mathfrak{t})$. Then for every $I' \in \mathcal{I}$, there exists $\nu_{I'} \in \mathcal{A}$ such that $\lambda_I t_{I'} = \nu_{I'} t_I$ holds and so $\lambda_I a_{I'} = \nu_{I'} a_I$. Since this equality holds for every $I' \in \mathcal{I}$, λ_I has a factor a_I. Hence $\lambda_I \in (a_I)$. $\square$

From the result above, the reduced minor of the matrix T with respect to $I \in \mathcal{I}$ is equal to the quotient ideal $((t_I) : \mathfrak{t})$ up to a unit multiple of $\mathcal{A}$ provided that $\mathcal{A}$ is a unique factorization domain.

Now that we have shown a new criterion (6.1) of the feedback stabilizability, in the following we present the relationship between generalized elementary factors and (6.1) by using radicals of ideals.

THEOREM 6.6. *Let Λ_{PI} denote the generalized elementary factor of the plant P with respect to I in $\mathcal{I}$. Then the radical of Λ_{PI} is equal to the radical of $((t_I) : \mathfrak{t})$.*

Before proving this result, we present an analogous result of Lemma 6.4.

LEMMA 6.7. *Let $\mathcal{R}$ be a commutative ring, $\mathfrak{a}, \mathfrak{b}$ ideals of $\mathcal{R}$, and $f \in \mathcal{R}$. Denote by $(\mathfrak{a} : \mathfrak{b})_f$ the localization of the quotient ideal $(\mathfrak{a} : \mathfrak{b})$ at f. Further*

let $(\mathfrak{a}_f : \mathfrak{b}_f)$ be the quotient ideal of $\mathcal{R}_f$, where $\mathfrak{a}_f$ and $\mathfrak{b}_f$ are localizations of principal ideals $\mathfrak{a}$ and $\mathfrak{b}$ at f, respectively. Then $(\mathfrak{a} : \mathfrak{b})_f = (\mathfrak{a}_f : \mathfrak{b}_f)$ holds.

Analogously to Lemma 6.4, the proof of this lemma is omitted.

Proof of Theorem 6.6. Let I be fixed. We first show (i) $\Lambda_{PI} \subset \sqrt{((t_I) : \mathfrak{t})}$ and then (ii) $\sqrt{\Lambda_{PI}} \supset ((t_I) : \mathfrak{t})$. They are sufficient to prove this theorem.

(i). Let λ be an arbitrary but fixed element of Λ_{PI}. Then there exists a matrix K over $\mathcal{A}$ with $\lambda T = K\Delta_I T$. Then for every $I' \in \mathcal{I}$, we have $\lambda\Delta_{I'}T = \Delta_{I'}K\Delta_I T$, so that $\lambda^m t_{I'} = \det(\Delta_{I'}K)t_I$. This implies $\lambda^m \in ((t_I) : (t_{I'}))$. Hence we have $\lambda^m \in \bigcap_{I'\in\mathcal{I}}((t_I) : (t_{I'})) = ((t_I) : \sum_{I'\in\mathcal{I}}(t_{I'}))$.

(ii). Let λ be an arbitrary but fixed element of $((t_I) : \mathfrak{t})$. Then $((t_I) : \mathfrak{t})_\lambda = \mathcal{A}_\lambda$ and hence $((t_I)_\lambda : \mathfrak{t}_\lambda) = \mathcal{A}_\lambda$ by Lemma 6.7. This implies that $(t_I)_\lambda = \mathfrak{t}_\lambda$ and further that every full-size minor of T has a factor t_I over $\mathcal{A}_\lambda$. Since t_I is a factor of $\det(D)$, it is a nonzerodivisor of $\mathcal{A}_\lambda$. Now let $T' = T(\mathrm{adj}(\Delta_I T))$ and $t'_{I'} = \det(\Delta_{I'}T')$ for every $I' \in \mathcal{I}$. Then $t'_{I'} = t_{I'}\det(\mathrm{adj}(\Delta_I T))$ and $\Delta_I T' = t_I E_m$ hold. Since $\det(\mathrm{adj}(\Delta_I T)) = t_I^{m-1}$, every $t'_{I'}$ has a common factor t_I^m.

Analogously to the proof of Theorem 5.2, we can show that every entry of T' has a factor t_I. Let $T'' = T'/t_I$ over $\mathcal{A}_\lambda$. Then $T = T''\Delta_I T$ holds over $\mathcal{A}_\lambda$. Hence there exists an integer ω such that $\lambda_I^\omega T''$ can be considered over $\mathcal{A}$ and further $\lambda_I^\omega T = \lambda_I^\omega T''\Delta_I T$ holds over $\mathcal{A}$. Now letting $K = \lambda_I^\omega T''\Delta_I$, we have that λ_I^ω is an element of Λ_{PI} and hence $\lambda_I \in \sqrt{\Lambda_{PI}}$. □

In the case where $\mathcal{A}$ is a unique factorization domain, we obtain the following result which connects Theorems 3.4 and 3.6.

THEOREM 6.8. *Suppose that $\mathcal{A}$ is a unique factorization domain. Let P be a causal plant and I in $\mathcal{I}$. Then the radical of the elementary factor of the matrix T with respect to I is equal to the radical of the reduced minor of T with respect to I up to a unit multiple.*

Proof. Let f_I denote the elementary factor of the matrix T with respect to I. Also let a_I denote the reduced minor of T with respect to I.

In the case where $\mathcal{A}$ is a unique factorization domain, the generalized elementary factor of the plant P with respect to I is equal to the principal ideal (f_I). Thus, by Theorem 6.6, $\sqrt{(f_I)} = \sqrt{((t_I) : \mathfrak{t})}$. By virtue of Proposition 6.5, we have $\sqrt{(f_I)} = \sqrt{(a_I)}$. □

7. Concluding Remarks. We have presented two generalization of the reduced minors. One is the full-size minor ideal. Its projectivity is a criterion of the feedback stabilizability(Theorem 5.2). The other is quotient ideals in (6.2). Their coprimeness is a criterion of the feedback stabilizability(Theorem 6.1).

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